Necessary and sufficient conditions for the existence of α-determinantal processes

Franck Maunoury

Necessary and sufficient conditions for the existence of α-determinantal processes

Franck Maunoury *

Submitted: June 2014 - Revised: April 2015

Abstract

We give necessary and sufficient conditions for existence and infinite divisibility of α-determinantal processes. For that purpose we use results on negative binomial and ordinary binomial multivariate distributions.

Keywords: determinantal process, permanental process, α-determinantal process, α-permanental process, infinitely divisible, complete monotonicity, fermion process, boson process.

AMS 2010 subject classifications: 60G55, 60E07.

1 Introduction

Several authors have already established necessary and sufficient conditions for existence of α-determinantal processes.
Macchi in [8] and Soshnikov in its survey paper [11] gave a necessary and sufficient condition for determinantal processes with self-adjoint kernels, which corresponds to the case $\alpha = -1$.
The same condition has also been established in a different way by Hough, Krishnapur, Peres and Virág in [7] in the case $\alpha = -1$. They have also given a sufficient condition of existence in the case $\alpha = 1$ and self-adjoint kernel.
In the special case when the configurations are on a finite space, the paper of Vere-Jones [12] provides necessary and sufficient conditions for any value of α.
Finally, Shirai and Takahashi have given sufficient conditions for the existence of an α-determinantal process for any values of α. However, in the case $\alpha > 0$, their sufficient condition (Condition B) in [9] does not work for the following example: the space is reduced to a single point space and the reference measure λ is a unit point mass. With their notations, the two kernels K and J_α are respectively reduced to two real numbers k and j_α, with

$$j_\alpha = \frac{k}{1 + \alpha k}$$

*Université Pierre et Marie Curie / Telecom ParisTech, France. E-mail: franck.maunoury@upmc.fr

We can choose $\alpha > 0$ and $k < 0$ such that $j_\alpha > 0$. Under these assumptions, Condition B is fulfilled but the obtained point process has a negative correlation function ($\rho_1(x) = k$), which has to be excluded, since a correlation function is an almost everywhere non-negative function.

We are going to strengthen Condition B of Shirai and Takahashi and obtain a necessary and sufficient condition in the case $\alpha > 0$. This is presented in Theorem 1.

Besides, in the case $\alpha < 0$, we extend the result of Shirai and Takahashi to the case of non self-adjoint kernels and show that the obtained condition is also necessary (Theorems 4 and 5). Moreover, we show that $-1/\alpha$ is necesserely an integer. This has been noticed by Vere-Jones in [13] in the case of configurations on a finite space.

We also give a necessary and sufficient condition for the infinite divisibility of an α-determinantal process for all values of α.

The main results are presented in Section 3. Section 2 introduces the needed notation. In Section 4, we write a multivariate version of a Shirai and Takahashi formulae on Fredholm determinant expansion. Sections 5 and 6 present the proofs of the results concerning respectively the cases $\alpha > 0$ and $\alpha < 0$. The proofs concerning infinite divisibility are presented in Section 7.

2 Preliminaries

Let E be a locally compact Polish space. A locally finite configuration on E is an integer-valued positive Radon measure on E. It can also be identified with a set $\{(M, \alpha_M) : M \in F\}$, where F is a countable subset of E with no accumulation points (i.e. a discrete subset of E) and, for each point in F, α_M is a non-null integer that corresponds to the multiplicity of the point M (M is a multiple point if $\alpha_M \geq 2$).

Let λ be a Radon measure on E. Let $\mathcal{X}$ be the space of the locally finite configurations of E. The space $\mathcal{X}$ is endowed with the vague topology of measures, i.e. the smallest topology such that, for every real continuous function f with compact support, defined on E , the mapping

$$\mathcal{X} \ni \xi \mapsto \langle f, \xi \rangle = \sum_{x \in \xi} f(x) = \int f \mathrm{d}\xi$$

is continuous. Details on the topology of the configuration space can be found in [1].

We denote by $\mathcal{B}(\mathcal{X})$ the corresponding σ-algebra. A point process on E is a random variable with values in $\mathcal{X}$. We do not restrict ourselves to simple point processes, as the configurations in $\mathcal{X}$ can have multiple points.

For a $n \times n$ matrix $A = (a_{[ij})_{1 \leq i,j \leq n}$, set:

$$\det_\alpha A = \sum_{\sigma \in \Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where Σ_n is the set of all permutations on $\{1, \ldots, n\}$ and $\nu(\sigma)$ is the number of cycles of the permutation σ.

For a relatively compact set $\Lambda \subset E$, the Janossy densities of a point process ξ w.r.t. a Radon measure λ are functions (when they exist) $j_n^\Lambda : E^n \to [0, \infty)$ for $n \in N$, such that

$$j_n^\Lambda(x_1, \ldots, x_n) = n!\, \mathbb{P}(\xi(\Lambda) = n)\, \pi_n^\Lambda(x_1, \ldots, x_n)$$
$$j_0^\Lambda(\emptyset) = \mathbb{P}(\xi(\Lambda) = 0),$$

where π_n^Λ is the density with respect to $\lambda^{\otimes n}$ of the ordered set $(x_1, \ldots, x_n)$, obtained by first sampling ξ, given that there are n points in Λ, then choosing uniformly an order between the points.
For $\Lambda_1, \ldots, \Lambda_n$ disjoint subsets included in Λ, $\int_{\Lambda_1 \times \cdots \times \Lambda_n} j_n^\Lambda(x_1, \ldots, x_n)\lambda(dx_1) \ldots \lambda(dx_n)$ is the probability that there is exactly one point in each subset Λ_i $(1 \leq i \leq n)$, and no other point elsewhere.
We recall that we have the following formula, for a non-negative mesurable function f with support in a relatively compact set $\Lambda \subset E$:

$$\mathbb{E}(f(\xi)) = f(\emptyset)\, j_0^\Lambda(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda^n} f(x_1, \ldots, x_n)\, j_n^\Lambda(x_1, \ldots, x_n)\lambda(dx_1) \ldots \lambda(dx_n).$$

For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, we denote $a^{(n)} = \prod_{i=0}^{n-1}(a - i)$.
The correlation functions (also called joint intensities) of a point process ξ w.r.t. a Radon measure λ are functions (when they exist) $\rho_n : E^n \to [0, \infty)$ for $n \geq 1$, such that for any family of mutually disjoint relatively compact subsets $\Lambda_1, \ldots, \Lambda_d$ of E and for any non-null integers $n_1, \ldots, n_d$ such that $n_1 + \cdots + n_d = n$, we have

$$\mathbb{E}\left(\prod_{i=1}^{d} \xi(\Lambda_i)^{(n_i)}\right) = \int_{\Lambda_1^{n_1} \times \cdots \times \Lambda_d^{n_d}} \rho_n(x_1, \ldots, x_n)\lambda(dx_1), \ldots, \lambda(dx_n).$$

Intuitively, for a simple point process, $\rho_n(x_1, \ldots, x_n)\lambda(dx_1) \ldots \lambda(dx_n)$ is the infinitesimal probability that there is at least one point in the vicinity of each x_i (each vicinity having an infinitesimal volume $\lambda(dx_i)$ around x_i), $1 \leq i \leq n$.

Let α be a real number and K a kernel from E^2 to $\mathbb{R}$ or $\mathbb{C}$. An α-determinantal point process, with kernel K with respect to λ (also called α-permanental point process) is defined, when it exists, as a point process with the following correlation functions $\rho_n, n \in \mathbb{N}$ with respect to λ:

$$\rho_n(x_1, \ldots, x_n) = \det{}_\alpha(K(x_i, x_j))_{1 \leq i,j \leq n}.$$

We denote by $\mu_{\alpha,K,\lambda}$ the probability distribution of such a point process.
We exclude the case of a point process almost surely reduced to the empty configuration.

The case $\alpha = -1$ corresponds to a determinantal process and the case $\alpha = 1$ to a permanental process. The case $\alpha = 0$ corresponds to the Poisson point process. We suppose in the following that $\alpha \neq 0$.

We will always assume that the kernel K defines a locally trace class integral operator $\mathcal{K}$ on $L^2(E, \lambda)$. Under this assumption, one obtains an equivalent definition for the α-determinantal process, using the following Laplace functional formula:

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[\exp\left(-\int_E f d\xi\right)\right] = \mathrm{Det}\left(\mathcal{I} + \alpha\mathcal{K}[1 - e^{-f}]\right)^{-1/\alpha} \tag{1}$$

where f is a compactly-supported non-negative function on E, $\mathcal{K}[1-e^{-f}]$ stands for $\sqrt{1-e^{-f}}\mathcal{K}\sqrt{1-e^{-f}}$, $\mathcal{I}$ is the identity operator on $L^2(E,\lambda)$ and Det is the Fredholm determinant. Details on the link between the correlation function and the Laplace functional of an α-determinantal process can be found in the chapter 4 of [9]. Some explanations and useful formula on the Fredholm determinant are given in chapter 2.1 of [9].

For a subset $\Lambda \subset E$, set: $\mathcal{K}_\Lambda = p_\Lambda \mathcal{K} p_\Lambda$, where p_Λ is the orthogonal projection operator from $L^2(E,\lambda)$ to the subspace $L^2(\Lambda,\lambda)$.

For two subsets $\Lambda, \Lambda' \subset E$, set: $\mathcal{K}_{\Lambda\Lambda'} = p_\Lambda \mathcal{K} p_{\Lambda'}$, and denote by $K_{\Lambda\Lambda'}$ its kernel. We have for any $x, y \in E$, $K_{\Lambda\Lambda'}(x,y) = \mathbb{1}_\Lambda(x)\,\mathbb{1}_{\Lambda'}(y)\,K(x,y)$.

When $\mathcal{I}+\alpha\mathcal{K}$ (resp. $\mathcal{I}+\alpha\mathcal{K}_\Lambda$) is invertible, $\mathcal{J}_\alpha$ (resp. $\mathcal{J}_\alpha^\Lambda$) is the integral operator defined by: $\mathcal{J}_\alpha = \mathcal{K}(\mathcal{I}+\alpha\mathcal{K})^{-1}$ (resp. $\mathcal{J}_\alpha^\Lambda = \mathcal{K}_\Lambda(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1}$) and we denote by J_α (resp. J_α^Λ) its kernel. Note that $\mathcal{J}_\alpha^\Lambda$ is not the orthogonal projection of $\mathcal{J}_\alpha$ on $L^2(\Lambda,\lambda)$.

3 Main results

Theorem 1. *For $\alpha > 0$, there exists an α-permanental process with kernel K iff:*

- $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) \geq 1$, *for any compact set $\Lambda \subset E$*
- $\det_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$, *for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n) \in \Lambda^n$.*

Remark 2. Even when E is a finite set, note that the second condition of Theorem 1 consists in an infinite number of computations. Finding a simpler condition, that could be checked in a finite number of steps is still an open problem.

Theorem 3. *For $\alpha > 0$, if an α-permanental process with kernel K exists, then:*

$$\operatorname{Spec}\mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2\alpha}\} \text{ , for any compact set } \Lambda \subset E.$$

We remark that this condition is equivalent to

$$\operatorname{Spec}\mathcal{J}_\alpha^\Lambda \subset \{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\} \text{ , for any compact set } \Lambda \subset E$$

Theorem 4. *For $\alpha < 0$ and $\mathcal{K}$ an integral operator such that $\mathcal{I}+\alpha\mathcal{K}_\Lambda$ is invertible, for any compact set $\Lambda \subset E$, an α-determinantal process with kernel K exists iff the two following conditions are fulfilled:*

(i) $-1/\alpha \in \mathbb{N}$

(ii) $\det(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$, *for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n) \in \Lambda^n$.*

The arguments developed in the proof of Theorem 4 shows that actually $(ii) \implies (i)$. Consequently, Condition (ii) is itself a necessary and sufficient condition. It also implies that $\mathrm{Det}(\mathcal{I} + \beta\mathcal{K}_\Lambda) > 0$ for any $\beta \in [\alpha, 0]$ and any compact $\Lambda \subset E$.

Theorem 5. *For $\alpha < 0$ and $\mathcal{K}$ an integral operator such that for some compact set $\Lambda_0 \subset E$, $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$ is not invertible, an α-determinantal process with kernel K exists iff:*

(i') $-1/\alpha \in \mathbb{N}$

(ii') $\det(J_\beta^\Lambda(x_i, x_j))_{1\le i,j\le n} \ge 0$, *for any $n \in \mathbb{N}$, any $\beta \in (\alpha, 0)$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1, \dots, x_n) \in \Lambda^n$.*

As in Theorem 4, we also have $(ii') \implies (i')$ and Condition (ii') is itself a necessary and sufficient condition.

Note that $\mathcal{I} + \alpha\mathcal{K}_{\Lambda_0}$ is not invertible if and only if there is almost surely at least one point in Λ_0.

Corollary 6. *For m a positive integer, the existence of a $(-1/m)$-determinantal process with kernel K is equivalent to the existence of a determinantal process with the kernel $\dfrac{K}{m}$.*

Corollary 7. *For $\alpha < 0$ and $\mathcal{K}$ a self-adjoint operator, an α-determinantal process with kernel K exists iff:*

- $-1/\alpha \in \mathbb{N}$
- $\mathrm{Spec}\,\mathcal{K} \subset [0, -1/\alpha]$

This result is well known in the case $\alpha = -1$ (see for example Hough, Krishnapur, Peres and Virág in [7]).
The sufficient part of this necessary and sufficient condition corresponds to condition A in [9] of Shirai and Takahashi.

Theorem 8. *For $\alpha < 0$, an α-determinantal process in never infinitely divisible.*

Theorem 9. *For $\alpha > 0$, an α-determinantal process is infinitely divisible iff*

- $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) \ge 1$, *for any compact set $\Lambda \subset E$*
- $\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \ge 0$, *for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and $\lambda^{\otimes n}$-a.e. $(x_1, \dots, x_n) \in \Lambda^n$.*

This theorem gives a more general condition for infinite-divisibility of an α-permanental process than the condition given by Shirai and Takahashi in [9].

Theorem 10. *For $\mathcal{K}$ a a real symmetric locally trace class operator and $\alpha > 0$, an α-permanental process is infinitely divisible iff*

- $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) \geq 1$, *for any compact set* $\Lambda \subset E$
- $J_\alpha^\Lambda(x_1,x_2)\dots J_\alpha^\Lambda(x_{n-1},x_n)J_\alpha^\Lambda(x_n,x_1) \geq 0$, *for any* $n \in \mathbb{N}$, *any compact set* $\Lambda \subset E$ *and* $\lambda^{\otimes n}$*-a.e.* $(x_1,\dots,x_n) \in \Lambda^n$.

Following Griffith and Milne's remark in [6], when an α-permanental process with kernel K exists and is infinitely divisible, we can replace J_Λ^α by $|J_\Lambda^\alpha|$ and obtain an α-permanental process with the same probability distribution.

Remark 11. In Theorem 1, 9 and 10 , the condition

$$\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) \geq 1, \text{ for any compact set } \Lambda \subset E$$

can be replaced by

$$\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) > 0, \text{ for any compact set } \Lambda \subset E.$$

4 Fredholm determinant expansion

In [9], Shirai and Takahashi have proved the following formula

$$\mathrm{Det}(\mathcal{I}-\alpha z\mathcal{K})^{-1/\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{E^n} \det{}_\alpha(K(x_i,x_j))_{1\leq i,j\leq n}\lambda(dx_1)\dots\lambda(dx_n) \tag{2}$$

for a trace class integral operator $\mathcal{K}$ with kernel K and for $z \in \mathbb{C}$ such that $\|\alpha z\mathcal{K}\| < 1$. In the case where the space E is finite, this formula is also given by Shirai in [10].

As $z \mapsto \mathrm{Det}(\mathcal{I}-\alpha z\mathcal{K})$ is analytic on $\mathbb{C}$ and $z \mapsto z^{-1/\alpha}$ is analytic on $\mathbb{C}^*$, we obtain that $z \mapsto \mathrm{Det}(\mathcal{I}-\alpha z\mathcal{K}_{\Lambda,\alpha})^{-1/\alpha}$ is analytic on $\{z \in \mathbb{C} : \mathcal{I}-\alpha z\mathcal{K}_{\Lambda,\alpha} \text{ invertible}\}$.
Therefore, the formula can be extended to the open disc D, centered in 0 with radius $R = \sup\{r \in \mathbb{R}_+ : \forall z \in \mathbb{C}, |z| < r \Rightarrow \mathcal{I}-\alpha z\mathcal{K} \text{ is invertible}\}$.
D is the open disc of center 0 and radius $1/\|\alpha\mathcal{K}\|$, if the operator $\mathcal{K}$ is self-adjoint, but it can be larger if $\mathcal{K}$ is not self-adjoint.

As remarked by Shirai and Takahashi, the formula (2) is valid for any $z \in \mathbb{C}$ if $-1/\alpha \in \mathbb{N}$.

The following proposition extends (2) to a multivariate case.

Proposition 12. *Let* $\Lambda \subset E$ *be a relatively compact set,* $\Lambda_1,\dots\Lambda_d$ *mutually disjoint subsets of* Λ *and* $\mathcal{K}$ *a locally trace class integral operator with kernel* K.
We have the following formula

$$\begin{aligned}&\mathrm{Det}\left(\mathcal{I}-\alpha\sum_{k=1}^{d} z_k\,\mathcal{K}_{\Lambda_k\Lambda}\right)^{-1/\alpha}\\ &\quad= \sum_{n_1,\dots,n_d=0}^{\infty}\left(\prod_{k=1}^{d}\frac{z_k^{n_k}}{n_k!}\right)\int_{\Lambda_1^{n_1}\times\dots\times\Lambda_d^{n_d}} \det{}_\alpha(K(x_i,x_j))_{1\leq i,j\leq n}\,\lambda(dx_1)\dots\lambda(dx_n)\end{aligned} \tag{3}$$

for any $z_1,\dots,z_d \in \mathbb{C}$, *such that* $\mathcal{I}-\alpha\gamma\sum_{k=1}^d z_k\mathcal{K}_{\Lambda_k\Lambda}$ *is invertible for any complex number* γ *satisfying* $|\gamma| < 1$ *(*n *denotes* $n_1+\dots+n_d$*).*

Proof. We apply the formula (2) to the class trace operator $\sum_{k=1}^{d} z_k \mathcal{K}_{\Lambda_k \Lambda}$ and we use the multilinearity property of the α-determinant of a matrix with respect to its rows.

We obtain

$$\begin{aligned}
&\mathrm{Det}\left(\mathcal{I} - \alpha \sum_{k=1}^{d} z_k \mathcal{K}_{\Lambda_k \Lambda}\right)^{-1/\alpha} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \det_\alpha \left(\sum_{k=1}^{d} z_k K_{\Lambda_k \Lambda}(x_i, x_j)\right)_{1\le i,j\le n} \lambda(dx_1)\ldots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{E^n} \sum_{k_1,\ldots k_n=1}^{d} \det_\alpha \left(z_{k_i} \mathbb{1}_{\Lambda_{k_i}}(x_i) \mathbb{1}_{\Lambda}(x_j) K(x_i, x_j)\right)_{1\le i,j\le n} \lambda(dx_1)\ldots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots k_n=1}^{d} \int_{\Lambda_{k_1}\times\cdots\times\Lambda_{k_n}} \det_\alpha \left(z_{k_i} K(x_i, x_j)\right)_{1\le i,j\le n} \lambda(dx_1)\ldots\lambda(dx_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1,\ldots k_n=1}^{d} \left(\prod_{i=1}^{n} z_{k_i}\right) \int_{\Lambda_{k_1}\times\cdots\times\Lambda_{k_n}} \det_\alpha \left(K(x_i, x_j)\right)_{1\le i,j\le n} \lambda(dx_1)\ldots\lambda(dx_n)
\end{aligned}$$

where we have used the fact that $K_{\Lambda_k \Lambda}(x_i, x_j) = \mathbb{1}_{\Lambda_k}(x_i)\, \mathbb{1}_{\Lambda}(x_j) K(x_i, x_j)$ for the equality between the first and the second line.
As the value of the α-determinant of a matrix is unchanged by simultaneous interchange of its rows and its columns, the product $z_1^{n_1}\ldots z_d^{n_d}$ where $n_1 + \ldots n_d = n$, will be repeated $\binom{n}{n_1 \ldots n_d}$ times. This gives the desired formula.

$\square$

For a relatively compact set $\Lambda \subset E$ and $\Lambda_1, \ldots, \Lambda_d$ mutually disjoint subsets of Λ, the computation of the Laplace functional of an α-determinantal process for the function $f : (z_1, \ldots, z_d) \mapsto -\sum_{k=1}^{d} (\log z_k) \mathbb{1}_{\Lambda_k}$, with $z_1, \ldots, z_d \in (0, 1]$ gives thanks to (1):

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[\prod_{k=1}^{d} z_k^{\xi(\Lambda_k)}\right] = \mathrm{Det}\left(\mathcal{I} + \alpha \sum_{k=1}^{d} (1 - z_k)\, \mathcal{K}_{\Lambda_k \Lambda}\right)^{-1/\alpha} \tag{4}$$

which is the probability generating function (p.g.f.) of the finite-dimensional random vector $(\xi(\Lambda_1), \ldots, \xi(\Lambda_d))$.

For $\alpha < 0$, the formula (4) reminds the multivariate binomial distribution p.g.f. and for $\alpha > 0$, the multivariate negative binomial distribution p.g.f., given by Vere-Jones in [12], in the special case where the space E is finite.

5 α- permanental process ($\alpha > 0$)

Proof of Theorem 1. We first prove that the conditions are necessary. We suppose that there exists an α-permanental process with $\alpha > 0$, kernel K defining the locally trace class integral operator $\mathcal{K}$.

By taking $d=1$ in the formula (4), we have

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left(z^{\xi(\Lambda)}\right) = \mathrm{Det}\left(\mathcal{I}+\alpha(1-z)\,\mathcal{K}_\Lambda\right)^{-1/\alpha}$$

for any compact set $\Lambda \subset E$ and $z \in (0,1]$.

Thus, $\mathrm{Det}(\mathcal{I}+\alpha(1-z)\mathcal{K}_\Lambda) \geq 1$ for $z \in (0,1]$. By continuity (as $z \mapsto \mathrm{Det}(\mathcal{I}+(1-z)\mathcal{K}_\Lambda)$ is indeed analytic on $\mathbb{C}$), we obtain that $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda) \geq 1$, which is the first condition. This implies that for any compact set $\Lambda \subset E$, $\mathcal{I}+\alpha\mathcal{K}_\Lambda$ is invertible. Hence $\mathcal{J}_\alpha^\Lambda$ exists and we have, for any non-negative function f, with compact support included in Λ

$$\begin{aligned}
&\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left(\prod_{x\in\xi} e^{-f(x)}\right) = \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}[1-e^{-f}])^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda(1-e^{-f}))^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{Det}(\mathcal{I}-\alpha\mathcal{J}_\alpha^\Lambda e^{-f})^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\left(\prod_{i=1}^{n} e^{-f(x_i)}\right)\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n}\lambda(dx_1)\ldots\lambda(dx_n)
\end{aligned} \tag{5}$$

where we have used for the equality between the first and the second line the fact that $\mathrm{Det}(\mathcal{I}+\mathcal{AB}) = \mathrm{Det}(\mathcal{I}+\mathcal{BA})$, for any trace class operator $\mathcal{A}$, and any bounded operator $\mathcal{B}$.
As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0 \;\; \lambda^{\otimes n}\text{-a.e. } (x_1,\ldots,x_n)\in E^n$$

$j_{\alpha,n}^\Lambda(x_1,\ldots,x_n) = \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n}$ is the Janossy density.

Conversely, if we assume $\mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$ and $\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$ for any $n \in \mathbb{N}$, any compact set $\Lambda \subset E$ and any $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$, the Janossy density will be correctly defined and, on any compact set Λ, we get the existence of a point process ξ_Λ with kernel K_Λ (see Proposition 5.3.II. in [2] - here the normalization condition is automatic by chosing $f=0$ in (5)).
The restriction of a point process η, defined on $\Lambda' \subset E$, to a subspace $\Lambda \subset \Lambda'$ is the point process denoted $\eta|_\Lambda$, obtained by keeping the points in Λ and deleting the points in $\Lambda'\backslash\Lambda$. For any compact sets $\Lambda, \Lambda' \subset E$, such that $\Lambda \subset \Lambda'$, ξ_Λ and $\xi_{\Lambda'}|_\Lambda$ have the same Laplace functional, because we have for any non-negative function f, with compact support included in Λ:

$$\begin{aligned}
\mathbb{E}\left(\exp\left(-\int_\Lambda f d\xi_{\Lambda'}|_\Lambda\right)\right) &= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_{\Lambda'}[1-e^{-f}])^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I}+\alpha\mathcal{K}_{\Lambda}[1-e^{-f}])^{-1/\alpha} \\
&= \mathbb{E}\left(\exp\left(-\int_\Lambda f d\xi_\Lambda\right)\right).
\end{aligned}$$

Therefore, ξ_Λ and $\xi_{\Lambda'}|_\Lambda$ have the same probability distribution. We say that the family $(\mathcal{L}(\xi_\Lambda))$, Λ compact set included in E, is consistent.

Then we can obtain a point process on the complete space E by the Kolmogorov existence theorem for point processes (see Theorem 9.2.X in [3] with $P_k(A_1, \dots, A_k; n_1, \dots, n_k) = \mathbb{P}\left(\xi_{\cup_{i=1}^k A_i}(A_1) = n_1, \dots, \xi_{\cup_{i=1}^k A_i}(A_k) = n_k\right)$: as $\xi_{\cup_{i=1}^k A_i}$ is a point process, it follows that the properties (i), (iii), (iv) are fulfilled ; (ii) is fulfilled because the family $(\mathcal{L}(\xi_\Lambda))$, Λ compact set included in E, is consistent).
As we used, in this second part of the proof, only the fact that $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} > 0$ (instead of $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha} \geq 1$), the assertion in remark 11 is also proved.

□

Proof of Theorem 3. We suppose there exists an α-permanental process with $\alpha > 0$, kernel K defining the locally trace class integral operator $\mathcal{K}$.

Then, following the proof of the preceding theorem, we get that, for all $z \in [0, 1]$

$$\mathrm{Det}(\mathcal{I} + \alpha(1 - z)\mathcal{K}_\Lambda) = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)\,\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda) > 0.$$

As the power series of $\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$ has all its terms non-negative,

$$|(\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\Lambda^\alpha)^{-1/\alpha}| \leq (\mathrm{Det}(\mathcal{I} - \alpha\,|z|\,\mathcal{J}_\Lambda^\alpha)^{-1/\alpha}.$$

If z_0 is a complex number with minimum modulus such that $(\mathrm{Det}(\mathcal{I} - \alpha z_0\mathcal{J}_\Lambda^\alpha) = 0$, by analycity of $z \mapsto \mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)$ on $\mathbb{C}$ and $z \mapsto z^{-1}$ on $\mathbb{C}^*$, $\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha}$ converges for $|z| < |z_0|$ and diverges for $z = z_0$. Thus the series diverges in $z = |z_0|$ and $|z_0| > 1$. This means that the series converges for $|z| \leq 1$ thus, in this case, $\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda) > 0$.

This implies the necessary condition: $\mathrm{Spec}\,\mathcal{J}_\alpha^\Lambda \subset \{z \in \mathbb{C} : |z| < \dfrac{1}{\alpha}\}$.

As ν eigenvalue of $\mathcal{K}$ is equivalent to $\dfrac{\nu}{1 + \alpha\nu}$ eigenvalue of $\mathcal{J}$, and as, $\mathcal{K}$ and $\mathcal{J}$ being compact operators, their non-null spectral values are their eigenvalues, we get the other equivalent necessary condition:

$$\mathrm{Spec}\,\mathcal{K}_\Lambda \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > -\frac{1}{2\alpha}\}.$$

□

6 α- determinantal process ($\alpha < 0$)

We recall the following remark, already made for example in [7].

Remark 13. If we define kernels only $\lambda^{\otimes 2}$-almost everywhere, there can be problems when we consider only the diagonal terms, as $\lambda^{\otimes 2}\{(x, x) : x \in \Lambda\} = 0$. For example, in the formula

$$\mathrm{tr}\, K_\Lambda = \int_\Lambda K(x, x)\lambda(dx),$$

$\mathrm{tr}\, K_\Lambda$ is not uniquely defined. To avoid this problem, we write the kernel K_Λ as follows:

$$K_\Lambda(x, y) = \sum_{k=0}^{\infty} a_k \varphi_k(x)\overline{\psi_k}(y)$$

where $(\varphi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ are orthonormal basis in $L^2(\Lambda,\lambda)$ and $(a_k)_{k\in\mathbb{N}}$ is a sequence of non-negative real number, which are the singular values of the operator $\mathcal{K}_\Lambda$.
The functions φ_k and ψ_k, $k\in\mathbb{N}$, are defined λ-almost everywhere, but this gives then a unique value for the expression of type

$$\int_{\Lambda^n} F(K(x_i,x_j)_{1\le i,j\le n})\, G(x_1,\dots,x_n)\lambda(dx_1)\dots\lambda(dx_n)$$

where F is an arbitrary complex function from $\mathbb{C}^{n^2}$ and G is an arbitrary complex function from Λ^n.
With this remark, the quantities that appear with $F=\det_\alpha$ are well defined.

Lemma 14. *Let K be a kernel defined as in Remark 13 and defining a trace class integral operator $\mathcal{K}$ on $L^2(\Lambda,\lambda)$, where Λ is a non-λ-null compact set included in the locally compact Polish space E, λ be a Radon measure, n an integer and α a real number. Let F be a continuous fonction from $\mathbb{C}^{n^2}$ to $\mathbb{C}$. The three following assertions are equivalent*

(i) $F(K(x_i,x_j)_{1\le i,j\le n})\ge 0$ $\lambda^{\otimes n}-a.e.(x_1,\dots,x_n)\in\Lambda^n$

(ii) there exists a set $\Lambda'\subset\Lambda$ such that $\lambda(\Lambda\backslash\Lambda')=0$ and $F((K(x_i,x_j))_{1\le i,j\le n})\ge 0$ for any $(x_1,\dots,x_n)\in(\Lambda')^n$

(iii) there exists a version of K such that $F((K(x_i,x_j))_{1\le i,j\le n})\ge 0$ for any $(x_1,\dots,x_n)\in\Lambda^n$

Proof. (i) is clearly a consequence of (ii). We assume now that (i) is satisfied and we denote by N the $\lambda^{\otimes n}$-null set of n-tuples $(x_1,\dots,x_n)\in\Lambda^n$ such that $F((K(x_i,x_j))_{1\le i,j\le n})<0$. As in remark 13, we write the kernel K as follows

$$K(x,y)=\sum_{k=0}^{\infty} a_k\varphi_k(x)\overline{\psi_k}(y)=\langle(\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}}(x)|(\sqrt{a_k}\psi_k)_{k\in\mathbb{N}}(y)\rangle$$

where $(\varphi_k)_{k\in\mathbb{N}}$, $(\psi_k)_{k\in\mathbb{N}}$ are orthonormal basis in $L^2(\Lambda,\lambda)$, $(a_k)_{k\in\mathbb{N}}$ is a sequence of non-negative real number, which are the singular values of the operator $\mathcal{K}$ and $\langle .|.\rangle$ denote the inner product in the Hilbert space $l_2(\mathbb{C})$.
As $\mathcal{K}$ is trace class, we have $\sum_{k=0}^{\infty} a_k<\infty$. Hence:

$$\sum_{k=0}^{\infty} a_k|\varphi_k(x)|^2<\infty \text{ and } \sum_{k=0}^{\infty} a_k|\psi_k(x)|^2<\infty\ \lambda\text{-a.e. } x\in\Lambda$$

From Lusin's theorem, there exists an increasing sequence $(A_p)_{p\in\mathbb{N}}$ of compact sets included in Λ such that, for any $p\in\mathbb{N}$

$$(\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}} \text{ and } (\sqrt{a_k}\psi_k)_{k\in\mathbb{N}} \text{ are continuous from } A_p \text{ to } l_2(\mathbb{C}) \text{ and } \lambda(\Lambda\backslash A_p)<\frac{1}{p}$$

Therefore the kernel $K:(x,y)\mapsto\left\langle(\sqrt{a_k}\varphi_k)_{k\in\mathbb{N}}(x)|(\sqrt{a_k}\psi_k)_{k\in\mathbb{N}}(y)\right\rangle$ is continuous on A_p^2.
As E is a Polish space, it can be endowed with a distance that we denote by d. We consider the sets

$$A'_p=\{x\in A_p:\forall r>0,\lambda(B(x,r)\cap A_p)>0\}$$
$$B_{p,n}=\{x\in A_p:\lambda(B(x,1/n)\cap A_p)=0\}$$

where $B(x,r)$ is the open ball in E of radius r centered at x and n is an integer.
Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $B_{p,n}$ converging to $x \in A_p$. Then we have, when $d(x,x_k) < 1/n$,

$$\lambda(B(x, 1/n - d(x,x_k) \cap A_p) \leq \lambda(B(x_k, 1/n) \cap A_p) = 0$$

Therefore $\lambda(B(x,1/n) \cap A_p) = 0$ and $x \in B_{p,n}$: $B_{p,n}$ is closed, thus compact (as it is included in the compact set A_p).
The set of open balls $\{B(x,1/n) : x \in B_{p,n}\}$ is a cover of $B_{p,n}$. Then, by compactness, $B_{p,n}$ can be covered by a finite numbers of such balls. As the intersections of A_p and any such a ball is a λ-null set, we get $\lambda(B_{p,n}) = 0$.

Hence we have: $\lambda(A'_p) = \lambda\left(A_p \backslash \cup_{n\in\mathbb{N}} B_{p,n}\right) = \lambda(A_p) > \lambda(\Lambda) - 1/p$.

Let $(x_1,\ldots,x_n) \in (A'_p)^n$. If $(x_1,\ldots,x_n) \notin N$, then $F((K(x_i,x_j))_{1\leq i,j\leq n}) \geq 0$.

Otherwise $(x_1,\ldots,x_n) \in N$. For any $i \in [\![1,n]\!]$ and any $r > 0$, we have

$\lambda(A_p \cap B(x_i,r)) > 0$, then $\lambda^{\otimes n}(A_p^n \cap B((x_1,\ldots,x_n),r)) = \lambda^{\otimes n}(\prod_{i=1}^{n}(A_p \cap B(x_i,r))) > 0$.

where $B((x_1,\ldots,x_n),r)$ denotes the open ball of radius r centered at x, in E^n endowed with the distance $d((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max_{1\leq i\leq n} d(x_i,y_i)$.
Then, as $\lambda^{\otimes n}(N) = 0$, for any $q \in \mathbb{N}$, there exists $(y_1^{(q)},\ldots,y_n^{(q)}) \in A_p^n \cap B((x_1,\ldots,x_n),1/q) \backslash N$ and thus $(y_1^{(q)},\ldots,y_n^{(q)})$ converge to $(x_1,\ldots,x_n)$ when $q \to \infty$.
As $(y_1^{(q)},\ldots,y_n^{(q)}) \notin N$, $F((K(y_i^{(q)},y_j^{(q)}))_{1\leq i,j\leq n}) \geq 0$.
As K is continuous on A_p^2 and F is continuous on $\mathbb{C}^{n^2}$, we have that the function $(x_1,\ldots,x_n) \mapsto F((K(x_i,x_j))_{1\leq i,j\leq n})$ is continuous on A_p^n. Hence we have: $F((K(x_i,x_j))_{1\leq i,j\leq n}) \geq 0$.
Therefore, in all cases, if $(x_1,\ldots,x_n) \in (A'_p)^n$, $F((K(x_i,x_j))_{1\leq i,j\leq n}) \geq 0$.

As $(A_p)_{p\in\mathbb{N}}$ is an increasing sequence, it is the same for $(A'_p)_{p\in\mathbb{N}}$. Hence we have: $\cup_{p\in\mathbb{N}}(A'_p)^n = \left(\cup_{p\in\mathbb{N}}A'_p\right)^n$.

We obtain:

$$F((K(x_i,x_j))_{1\leq i,j\leq n}) \geq 0 \text{ for any } (x_1,\ldots x_n) \in \left(\cup_{p\in\mathbb{N}}A'_p\right)^n$$

As $\lambda(\Lambda\backslash\left(\cup_{p\in\mathbb{N}}A'_p\right)) = 0$, we finally obtain (ii) with $\Lambda' = \cup_{p\in\mathbb{N}}A'_p$.

We obtained that (i) and (ii) are equivalent conditions.

(i) is clearly a consequence of (iii). Assume now (ii). We will define a version K_1 of K satisfying the condition (iii).
As $\lambda(\Lambda) \neq 0$, $\Lambda' \neq \emptyset$. We set an arbitrary $x_0 \in \Lambda'$.
For $(x,x') \in \Lambda^2$, we define, $y = x$ if $x \in \Lambda'$, $y = x_0$ if $x \in \Lambda\backslash\Lambda'$, $y' = x'$ if $x' \in \Lambda'$, $y' = x_0$ if $x' \in \Lambda\backslash\Lambda'$ and $K_1(x,x') = K(y,y')$.

For $(x_1,\dots,x_n)\in\Lambda^n$, we define, for $1\le i\le n$, $y_i=x_i$ if $x_i\in\Lambda'$ and $y_i=x_0$ if $x_i\in\Lambda\backslash\Lambda'$. Then we have, $F((K_1(x_i,x_j))_{1\le i,j\le n})=F((K(y_i,y_j))_{1\le i,j\le n})\ge 0$ and K_1 is a version of K satisfying the condition (iii).

□

Remark 15. Let $F_n, n\in\mathbb{N}$, be continuous functions from $\mathbb{C}^{n^2}$ to $\mathbb{C}$. For any non-$\lambda-null$ compact set Λ, the condition:

(i) $F_n((J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n})\ge 0$, for any $n\in\mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1,\dots,x_n)\in\Lambda^n$

can always be replaced by the equivalent conditions:

(ii) there exists a set $\Lambda'\subset\Lambda$ such that $\lambda(\Lambda\backslash\Lambda')=0$ and $F_n((J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n})\ge 0$, for any $n\in\mathbb{N}$ and $(x_1,\dots,x_n)\in(\Lambda')^n$.

or:

(iii) there exists a version of the kernel J such that $F_n((J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n})\ge 0$, for any $n\in\mathbb{N}$ and $(x_1,\dots,x_n)\in\Lambda^n$.

Proof. The proof of (ii) $\implies$ (iii) is done in the same way as in Lemma 14. The other parts of the proof are a direct application of Lemma 14. □

Proof that (i) is necessary in Theorem 4. This has been mentioned by Vere-Jones in [12] for the multivariate binomial probability distribution, which corresponds to a determinantal process with E being finite. To our knowledge, this has not been proved in other cases.
We consider the $n\times n$ matrix 1_n, whose elements are all equal to one.

We have: $\prod_{j=0}^{n-1}(1+j\alpha)=1+\sum_{k=1}^{n-1}\sum_{1\le j_1<\cdots<j_k\le n-1} j_1\dots j_k\ \alpha^k$

We will show by induction on n that the number of permutations in Σ_n having $n-k$ cycles for $k\neq 0$ is $a_{nk}=\sum_{1\le j_1<\cdots<j_k\le n-1} j_1\dots j_k$: this is true for $n=2$ and $k=1$. Assume it is true for a given $n\in\mathbb{N}^*$ and for any $k\in[\![1,n-1]\!]$. If we consider the permutations $\sigma\in\Sigma_{n+1}$ having $n+1-k$ cycles $(0\le k\le n)$, we have 2 cases:
- either $\sigma(n+1)=n+1$: there is exactly a_{nk} permutations corresponding to this case (with the convention $a_{nn}=0$, for the case $k=n$),
- or $\sigma(n+1)\neq n+1$. Then, if we denote $\tau_{n+1\,\sigma(n+1)}$ the transposition in Σ_{n+1} that exchange $n+1$ and $\sigma(n+1)$, $\tau_{n+1\,\sigma(n+1)}\circ\sigma$ is a permutation having $n+1$ as fixed point and $n+1-k$ other cycles (with elements in $[\![1,n]\!]$): there is exactly $na_{n\,k-1}$ permutations corresponding to this case.
Then we have

$$\begin{aligned} a_{n+1\;n+1-k} &= a_{nk}+na_{n\,k-1} \\ &= \sum_{1\le j_1<\cdots<j_k\le n-1} j_1\dots j_k + \sum_{\substack{1\le j_1<\cdots<j_{k-1}\le n-1\\ j_k=n}} j_1\dots j_k \\ &= \sum_{1\le j_1<\cdots<j_k\le n} j_1\dots j_k \end{aligned}$$

which is what we expected.
Thus: $\det_\alpha 1_n = \prod_{j=0}^{n-1}(1+j\alpha)$.
If $\alpha < 0$ but $-1/\alpha \notin \mathbb{N}$, there exists therefore $n \in \mathbb{N}$ such that $\det_\alpha 1_n < 0$.

We suppose that there exists an α-determinantal process with $\alpha < 0$ but $-1/\alpha \notin \mathbb{N}$ and kernel K. Then we have $\det_\alpha(K(x_i,x_j))_{1\le i,j\le n} \ge 0$ $\lambda^{\otimes n}$ -a.e. $(x_1\dots,x_n) \in E^n$.
As we exclude the case of a point process having no point almost surely and there is a sequence of compact sets Λ_p such that $\cup_{p\in\mathbb{N}}\Lambda_p = E$, there exists a compact set $\Lambda \in E$ such that

$$\mathbb{E}(\xi(\Lambda)) = \int_\Lambda K(x,x)\lambda(dx) > 0.$$

Applying Lemma 14, we get that there exist a version K_1 of the kernel K such that $\det_\alpha(K_1(x_i,x_j))_{1\le i,j\le n} \ge 0$ for any $(x_1\dots,x_n) \in \Lambda^n$. We also have:

$$\int_\Lambda K(x,x)\lambda(dx) = \int_\Lambda K_1(x,x)\lambda(dx) > 0.$$

Hence there exists $x_0 \in \Lambda$ such that $K_1(x_0,x_0) > 0$.
For $(x_1,\dots,x_n) = (x_0,\dots,x_0)$, we get:

$$\det_\alpha(K_1(x_i,x_j))_{1\le i,j\le n} = K(x_0,x_0)^n \det_\alpha 1_n < 0$$

which is a contradiction. Therefore if $\alpha < 0$ and an α-determinantal process exists, then α must be in $\{-1/m : m \in \mathbb{N}\}$.

□

We consider a $d\times d$ square matrix A. If $n_1,\dots,n_d$ are d non-negative integers, $A[n_1,\dots,n_d]$ is the $(n_1+\dots+n_d)\times(n_1+\dots+n_d)$ square matrix composed of the block matrices A_{ij}:

$$A[n_1,\dots,n_d] = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \dots & A_{dd} \end{pmatrix},$$

where A_{ij} is the $n_i \times n_j$ matrix whose elements are all equal to a_{ij} $(1 \le i,j \le d)$.

Lemma 16. *Given a $d\times d$ square matrix A, the following assertions are equivalent*

(i) $\det_{-1/m} A[n_1,\dots,n_d] \ge 0$, $\forall n_1,\dots,n_d \in \mathbb{N}$
(ii) $\det_{-1/m} A[n_1,\dots,n_d] \ge 0$, $\forall n_1,\dots,n_d \in \{0,\dots,m\}$
(iii) $\det A[n_1,\dots,n_d] \ge 0$, $\forall n_1,\dots,n_d \in \mathbb{N}$
(iv) $\det A[n_1,\dots,n_d] \ge 0$, $\forall n_1,\dots,n_d \in \{0,1\}$

Proof. If there exists $k \in [\![1,d]\!]$ such that $n_k > 1$, the matrix $A[n_1,\dots,n_d]$ has at least two identical rows and its determinant is null. So it is clear that (iii) and (iv) are equivalent.

We have:

$$\det(I+ZA)^m = \sum_{n_1,\dots,n_d=0}^{\infty} m^{n_1+\dots n_d} \left(\prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!}\right) \det_{-1/m} A[n_1,\dots,n_d] \tag{6}$$

where $Z = \mathrm{diag}(z_1,\dots,z_d)$ and $z_1,\dots,z_d$ are d complex numbers. It is a special case of the formula (3) with $\alpha = -1/m$, finite space $E = [\![1,d]\!]$ and reference measure λ atomic, where each point of E has measure 1, $\Lambda_k = \{k\}$, for $k \in [\![1,d]\!]$, $\Lambda = E$. Indeed, $ZA = \sum_{k=1}^d z_k A_k$, where A_k is the $d \times d$ square matrix having the same k^{th} row as A and the other rows with all elements equal to 0. The matrix A corresponds to the operator $\mathcal{K}$, the matrix A_k corresponds to the operator $\mathcal{K}_{\Lambda_k\Lambda}$. Formula (6) also corresponds to the one given by Vere-Jones in [13].

We also have for $m=1$:

$$\det(I+ZA) = \sum_{n_1,\dots,n_d=0}^{1} \left(\prod_{k=1}^{d} \frac{z_k^{n_k}}{n_k!}\right) \det A[n_1,\dots,n_d]. \tag{7}$$

as $\det A[n_1,\dots,n_d] = 0$ if there exists $k \in [\![1,d]\!]$ such that $n_k > 1$.

(i) is equivalent to the fact that the multivariate power series (6) has all its coefficients non-negative.
(iii) is equivalent to the fact that the multivariate power series (7) has all its coefficients non-negative.

The power series (6) being the m^{th} power of the power serie (7), if there exists $k \in [\![1,d]\!]$ such that $n_k > m$, the coefficient of $\prod_{k=1}^d z^{n_k}$ is null. Therefore, (i) is equivalent to (ii).

For the same reason, we also have that (i) is a consequence of (iii).

Conversely, following Vere-Jones in [12], we can show by induction on the order of the matrix A, that the fact that the power series (6) has all its coefficients non-negative implies that the power series (7) has all its coefficient non negative.
This proves the equivalence between (i) and (iii).

□

Proposition 17. *Let $\alpha < 0$ and $\mathcal{K}$ be an integral operator such that $\mathcal{I} + \alpha\mathcal{K}_\Lambda$ is invertible, for any compact set $\Lambda \subset E$. An α-determinantal process with kernel K exists iff:*

$$\det_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n \in \mathbb{N}, \text{ and any compact set } \Lambda \quad \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n) \in \Lambda^n \tag{8}$$

Condition (8) implies that $-\dfrac{1}{\alpha} \in \mathbb{N}$ and $\mathrm{Det}(\mathcal{I} + \beta\mathcal{K}) > 0$ for any $\beta \in [\alpha, 0]$.

Proof. We assume that there exists an α-determinantal process ξ with kernel K.
We already proved that it is necessary to have $-1/\alpha \in \mathbb{N}$.

By taking $d = 1$ in the formula (4), we have

$$\mathbb{E}\left(z^{\xi(\Lambda)}\right) = \mathrm{Det}\left(\mathcal{I} + \alpha(1-z)\,\mathcal{K}_\Lambda\right)^{-1/\alpha}$$

for any compact set $\Lambda \subset E$ and $z \in (0,1]$.

Then $\mathrm{Det}\left(\mathcal{I} + \alpha(1-z)\,\mathcal{K}_\Lambda\right) > 0$ for $z \in (0,1]$, and by continuity, $\mathrm{Det}\left(\mathcal{I} + \alpha\,\mathcal{K}_\Lambda\right) \geq 0$. As we assumed that $\mathcal{I} + \alpha K_\Lambda$ is invertible, we have necessarily $\mathrm{Det}\left(\mathcal{I} + \alpha\,\mathcal{K}_\Lambda\right) > 0$.

For any non-negative function f, with compact support included in Λ

$$\begin{aligned}
&\mathbb{E}\left(\prod_{x\in\xi} e^{-f(x)}\right) = \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}[1-e^{-f}])^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}\,\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda e^{-f})^{-1/\alpha} \\
&= \mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda)^{-1/\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\left(\prod_{i=1}^{n} e^{-f(x_i)}\right)\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n}\lambda(dx_1)\ldots\lambda(dx_n)
\end{aligned}$$

As the Laplace functional defines a.e. uniquely the Janossy density of a point process, one obtains:

$$\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0 \;\; \lambda^{\otimes n}\text{-a.e. } (x_1,\ldots,x_n) \in E^n$$

Conversely, we assume that the condition

> $\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0$, for any $n \in \mathbb{N}$, $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$ and any compact set Λ.

is fulfilled. We have

$$\mathrm{Det}(\mathcal{I} - \alpha z\mathcal{J}_\alpha^\Lambda)^{-1/\alpha} = \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{\Lambda^n}\mathrm{det}_\alpha(J_\alpha^\Lambda(x_i,x_j))_{1\leq i,j\leq n}\lambda(dx_1)\ldots\lambda(dx_n)$$

As $-1/\alpha \in \mathbb{N}$, this formula is valid for any $z \in \mathbb{C}$. Then we obtain for $z = 1$, $\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda)^{-1/\alpha} \geq 0$.
We also have $(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda)(\mathcal{I} + \alpha\mathcal{K}_\Lambda) = (\mathcal{I} + \alpha\mathcal{K}_\Lambda)(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda) = \mathcal{I}$.
Then $\mathrm{Det}(\mathcal{I} - \alpha\mathcal{J}_\alpha^\Lambda) > 0$ and $\mathrm{Det}(\mathcal{I} + \alpha\mathcal{K}_\Lambda) > 0$.
Thus the Janossy density is correctly defined and, on any compact set Λ we get the existence of a point process with kernel K and reference mesure λ.
Then it can be extended to the complete space E by the Kolmogorov existence theorem (see Theorem 9.2.X in [3]).

□

Proof of Theorem 4. For any $m \in \mathbb{N}$, applying Lemma 16, we have for any compact set Λ

$$\mathrm{det}_{-1/m}(J_{-1/m}^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1,\ldots,x_n) \in \Lambda^n$$

is equivalent to

$$\mathrm{det}(J_{-1/m}^\Lambda(x_i,x_j))_{1\leq i,j\leq n} \geq 0, \text{ for any } n \in \mathbb{N}, \text{ and any } (x_1,\ldots,x_n) \in \Lambda^n$$

Now, assume we only have

$$\det_{-1/m}(J^{\Lambda}_{-1/m}(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n\in\mathbb{N},\ \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n)\in\Lambda^n.$$

By lemma 14, for each $n\in\mathbb{N}$, there exists a set $\Lambda'_n \subset \Lambda$ such that $\lambda(\Lambda\backslash\Lambda'_n)=0$ and $\det_{-1/m}(J^{\Lambda}_{-1/m}(x_i,x_j))_{1\le i,j\le n}\ge 0$ for any $(x_1\dots,x_n)\in(\Lambda'_n)^n$.
If $\Lambda'=\cap_{n\in\mathbb{N}}\Lambda'_n$, we have $\lambda(\Lambda\backslash\Lambda')=0$ and $\det_{-1/m}(J^{\Lambda}_{-1/m}(x_i,x_j))_{1\le i,j\le n}\ge 0$ for any $n\in\mathbb{N}$ and $(x_1\dots,x_n)\in(\Lambda')^n$.
Then, by Lemma 16, we have: $\det(J^{\Lambda}_{-1/m}(x_i,x_j))_{1\le i,j\le n}\ge 0$, for any $n\in\mathbb{N}$ and $(x_1\dots,x_n)\in(\Lambda')^n$.
Therefore, we have

$$\det(J^{\Lambda}_{-1/m}(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n\in\mathbb{N},\ \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n)\in\Lambda^n.$$

The converse is done through a similar proof, using Lemma 14 and 16.
Thus, we obtain:

$$\det_{\alpha}(J^{\Lambda}_{\alpha}(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n\in\mathbb{N},\ \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n)\in\Lambda^n$$

is equivalent to

$$\det(J^{\Lambda}_{\alpha}(x_i,x_j))_{1\le i,j\le n} \ge 0, \text{ for any } n\in\mathbb{N},\ \lambda^{\otimes n}\text{-a.e. } (x_1,\dots,x_n)\in\Lambda^n$$

Theorem 4 is then a consequence of Proposition 17.

□

Proof of Theorem 5. We assume that there exists ξ an α-determinantal process with kernel K.
For $p\in(0,1)$, let ξ_p be the process obtained by first sampling ξ, then independently deleting each point of ξ with probability $1-p$.
Computing the correlation functions, we obtain that ξ_p is an α-determinantal process with kernel pK.
Thus we get from Theorem 4 that the conditions of the theorem must be fulfilled.

Conversely, we assume that these conditions are fulfilled. We obtain from Theorem 4 that an α-determinantal process ξ_p with kernel pK exists, for any $p\in(0,1)$.
We consider a sequence $(p_k)\in(0,1)^{\mathbb{N}}$ converging to 1 and a compact Λ.

$$\mathbb{E}(\exp(-t\xi_{p_k}(\Lambda)) = \mathrm{Det}(\mathcal{I}+\alpha p_k K_{\Lambda}(1-e^{-t}))^{-1/\alpha} \underset{k\to\infty}{\longrightarrow} \mathrm{Det}(\mathcal{I}+\alpha K_{\Lambda}(1-e^{-t}))^{-1/\alpha}$$

As $t\mapsto \mathrm{Det}(\mathcal{I}+\alpha K_{\Lambda}(1-e^{-t}))^{-1/\alpha}$ is continuous in 0, $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k\in\mathbb{N}}$ converge weakly. Thus $(\mathcal{L}(\xi_{p_k}(\Lambda)))_{k\in\mathbb{N}}$ is tight.
$\Gamma\subset\mathcal{X}$ is relatively compact if and only if, for any compact set $\Lambda\subset E$, $\{\xi(\Lambda):\xi\in\Gamma\}$ is bounded.
Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact sets such that $\cup_{n\in\mathbb{N}}\Lambda_n=E$.
As, for any $n\in\mathbb{N}$, $(\mathcal{L}(\xi_{p_k}(\Lambda_n)))_{k\in\mathbb{N}}$ is tight, we have that, for any $\epsilon>0$ and $n\in\mathbb{N}$, there exists $M_n>0$ such that for any $k\in\mathbb{N},\mathbb{P}(\xi_{p_k}(\Lambda_n)>M_n)<\epsilon\,2^{-n-1}$
Let $\Gamma=\{\gamma\in\mathcal{X}:\forall n\in\mathbb{N},\gamma(\Lambda_n)\le M_n\}$. It is a compact set and for any $k\in\mathbb{N},\mathbb{P}(\xi_{p_k}\in\Gamma^c)<\epsilon$.

Therefore, $(\mathcal{L}(\xi_{p_k}))_{k\in\mathbb{N}}$ is tight. As E is Polish, $\mathcal{X}$ is also Polish (endowed with the Prokhorov metric). Thus there is a subsequence of $(\mathcal{L}(\xi_{p_k}))_{k\in\mathbb{N}}$ converging weakly to the probability distribution of a point process ξ. By unicity of the distribution of an α-determinantal process for given kernel and reference measure, ξ must be an α-determinantal process with kernel K, which gives the existence.

□

Lemma 18. *Let $\mathcal{J}$ be a trace class self-adjoint integral operator with kernel J. We have*

$$\det(J(x_i,x_j))_{1\le i,j\le n}\ge 0 \textit{ for any } n\in\mathbb{N}, \lambda^{\otimes n}\textit{-a.e.}(x_1,\dots,x_n)\in\Lambda^n$$

if and only if

$$\operatorname{Spec}\mathcal{J}\subset[0,\infty)$$

Proof. If we assume that the operator $\mathcal{J}$ is positive, the kernel can be written as follows:

$$J(x,y)=\sum_{k=0}^{\infty}a_k\varphi_k(x)\overline{\varphi_k}(y)$$

where $a_k\ge 0$ for $k\in\mathbb{N}$.

Hence:

$$\det(J(x_i,x_j))_{1\le i,j\le n}\ge 0 \text{ for any } n\in\mathbb{N}, \text{ and any } (x_1,\dots,x_n)\in\Lambda^n$$

Conversely, assume that

$$\det(J(x_i,x_j))_{1\le i,j\le n}\ge 0 \text{ for any } n\in\mathbb{N}, \lambda^{\otimes n}\text{-a.e.}(x_1,\dots,x_n)\in\Lambda^n.$$

From formula (2) with $\alpha=-1$, we have then for any z $\in\mathbb{C}$

$$\operatorname{Det}(\mathcal{I}+z\mathcal{J})=\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{E^n}\det(J(x_i,x_j))_{1\le i,j\le n}\lambda(dx_1)\dots\lambda(dx_n). \tag{9}$$

As $\mathcal{J}$ is assumed to be self-adjoint, its spectrum is included in $\mathbb{R}$. Thanks to (9), it is impossible to have an eigenvalue in $\mathbb{R}_-^*$, as the power series has all its coefficients real non-negative and the first coefficient $(n=0)$ is real positive. Hence $\operatorname{Spec}\mathcal{J}\subset[0,\infty)$.

□

Proof of Corollary 7. We assume: $-1/\alpha\in\mathbb{N}$ and $\operatorname{Spec}\mathcal{K}\subset[0,-1/\alpha]$. Then we have, as $\mathcal{K}$ is self-adjoint, that for any compact set Λ, $\operatorname{Spec}\mathcal{K}_\Lambda\subset[0,-1/\alpha]$. Then $\operatorname{Det}(\mathcal{I}+\beta\mathcal{K}_\Lambda)>0$ for any $\beta\in(\alpha,0]$.
If $\mathcal{I}+\alpha K_\Lambda$ is invertible for any compact set $\Lambda\subset E$, we have $\operatorname{Spec}J_\alpha^\Lambda\subset[0,\infty)$ and J_α^Λ is a trace class self adjoint operator for any compact set Λ.
Then, applying Lemma 18, we get that

$$\det(J(x_i,x_j))_{1\le i,j\le n}\ge 0 \text{ for any } n\in\mathbb{N}, \text{ compact set } \Lambda \text{ and } \lambda^{\otimes n}\text{-a.e.}(x_1,\dots,x_n)\in\Lambda^n$$

Using Theorem 4, we get the existence of an α-determinantal process with kernel K.

When there exists a compact set Λ_0 such that $\mathcal{I}+\alpha K_{\Lambda_0}$ is not invertible, by the same line of proof, we obtain the announced result, using Theorem 5.

Conversely, we assume that there exists an α-determinantal process with kernel K. Then, from Theorem 4 or 5, we get that $-1/\alpha \in \mathbb{N}$.
If $\mathcal{I}+\alpha K_\Lambda$ is invertible for any compact set $\Lambda \subset E$, we have $\operatorname{Spec} J_\alpha^\Lambda \subset [0,\infty)$, using Theorem 4 and lemma 18. Then $\operatorname{Spec} K_\Lambda \subset [0,-1/\alpha) \subset [0,-1/\alpha]$, for any compact set Λ.
If there exists a compact set Λ_0 such that $\mathcal{I}+\alpha K_{\Lambda_0}$ is not invertible, we have $\operatorname{Spec} J_\beta^\Lambda \subset [0,\infty)$ for any compact set Λ and any $\beta \in (\alpha,0)$, using Theorem 5 and lemma 18. Then $\operatorname{Spec} K_\Lambda \subset [0,-1/\beta)$ for any $\beta \in (\alpha,0)$. Therefore $\operatorname{Spec} K_\Lambda \subset [0,-1/\alpha]$ for any compact set Λ.
As K is self-adjoint, this implies in both cases that $\operatorname{Spec} K \subset [0,-1/\alpha]$.

□

Remark 19. Using the known result in the case $\alpha=-1$ (see for example Hough, Krishnapur, Peres and Virág in [7]) and corollary 6, one obtains a direct proof of Corollary 7.

7 Infinite divisibility

Proof of Theorem 8. For $\alpha<0$, we have proved that it is necessary to have $-1/\alpha \in \mathbb{N}$. If an α-determinantal process was infinitely divisible, with $\alpha<0$, it would be the sum of N i.i.d αN-determinantal processes for any $N \in \mathbb{N}^*$, as it can be seen for the Laplace functional formula (1). This would imply that $-1/(N\alpha) \in \mathbb{N}$, for any $N \in \mathbb{N}^*$, which is not possible. Therefore, an α-determinantal process with $\alpha<0$ is never infinitely divisible. □

Some charactization on infinite divisibility have also been given in [4] in the case $\alpha>0$.

Proof of Theorem 9. For $\alpha>0$, assume that $\operatorname{Det}(\mathcal{I}+\alpha K_\Lambda) \geq 1$ and

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1} \prod_{i=1}^n J_\alpha^\Lambda(x_i,x_{\sigma(i)}) \geq 0,$$

for any compact set $\Lambda \subset E$, $n \in \mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$. Then we have:

$$\begin{aligned}\sum_{\sigma\in\Sigma_n:\nu(\sigma)=k} \prod_{i=1}^n J_\alpha^\Lambda(x_i,x_{\sigma(i)}) &= \sum_{\substack{\{I_1,\ldots,I_k\}\\ \text{partition of } [\![1,n]\!]}} \sum_{\substack{\sigma_1\in\Sigma(I_1),\ldots,\sigma_k\in\Sigma(I_k):\\ \nu(\sigma_1)=\cdots=\nu(\sigma_k)=1}} \prod_{q=1}^k \prod_{i\in I_q} J_\alpha^\Lambda(x_i,x_{\sigma_q(i)})\\ &= \sum_{\substack{\{I_1,\ldots,I_k\}\\ \text{partition of } [\![1,n]\!]}} \prod_{q=1}^k \left(\sum_{\substack{\sigma\in\Sigma(I_q):\\ \nu(\sigma)=1}} \prod_{i\in I_q} J_\alpha^\Lambda(x_i,x_{\sigma(i)}) \right) \geq 0,\end{aligned}$$

for any compact set $\Lambda \subset E$, $n \in \mathbb{N}$, $k \in [\![1,n]\!]$ and $\lambda^{\otimes n}$-a.e. $(x_1,\ldots,x_n) \in \Lambda^n$, where, for a finite set I, $\Sigma(I)$ denotes the set of all permutations on I.
Then, for any $N \in \mathbb{N}^*$ and any compact set $\Lambda \in E$, $\det_{N\alpha}(J_\alpha^\Lambda(x_i,x_j)/N)_{1\leq i,j\leq n} \geq 0$. From Theorem 1, we get that there exists a $(N\alpha)$-permanental process with kernel K/N. This

means that an α-permanental process with kernel K is infinitely divisible.

Conversely, if we assume an α-permanental process with kernel K is infinitely divisible, we get the existence of a $N\alpha$-permanental process with kernel K/N, for any $N \in \mathbb{N}^*$. From Theorem 1, we have that $\mathrm{Det}(\mathcal{I} + \alpha K_\Lambda) \geq 1$ for any compact set $\Lambda \in E$.
We also have

$$\frac{1}{(N\alpha)^{n-1}} \det_{N\alpha}(J_\alpha^\Lambda(x_i, x_j))_{1\leq i,j\leq n} \geq 0,$$

for any $N \in \mathbb{N}^*$, any $n \in \mathbb{N}$, any compact set $\Lambda \in E$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.
When N tends to ∞, we obtain:

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1} \prod_{i=1}^{n} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

which is the desired result.

$\square$

Proof of Theorem 10. We use the argument of Griffiths in [5] and Griffiths and Milne in [6]. Assume

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1} \prod_{i=1}^{n} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any $n \in \mathbb{N}$ and any $(x_1, \ldots, x_n) \in \Lambda^n$.
The condition $J_\alpha^\Lambda(x_1, x_2) \ldots J_\alpha^\Lambda(x_{n-1}, x_n) J_\alpha^\Lambda(x_n, x_1) \geq 0$ is satisfied for the elementary cycles, i.e. cycles such that $J_\alpha^\Lambda(x_i, x_j) = 0$ if $i < j + 1$ and ($i \neq 1$ or $j \neq n$). Then it can be extended to any cycle by induction, using $J_\alpha^\Lambda(x_i, x_j) = J_\alpha^\Lambda(x_j, x_i)$.

With Lemma 14, we can then extend the proof to the case when

$$\sum_{\sigma\in\Sigma_n:\nu(\sigma)=1} \prod_{i=1}^{n} J_\alpha^\Lambda(x_i, x_{\sigma(i)}) \geq 0,$$

for any $n \in \mathbb{N}$ and $\lambda^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in \Lambda^n$.

$\square$

Remark 20. Note that the argument from Griffiths and Milne in [5] and [6] is only valid for real symmetric matrices.

Acknowledgements

I express sincere thanks to Nathalie Eisenbaum for her support in the writing of this paper. I also thank the referee for his valuable comments that helped in improving the exposition.

References

[1] S. Albeverio, Y. G. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces. J. Funct. Anal. **154**, no. 2, 444–500 (1998)

[2] D. J. Daley, D. Vere-Jones, *An introduction to the theory of point processes - Vol. I: Elementary theory and methods*, 2^{nd} ed. (Springer, New York, 2003)

[3] D. J. Daley, D. Vere-Jones, *An introduction to the theory of point processes - Vol. II: General theory and structures*, 2^{nd} ed. (Springer, New York, 2008)

[4] N. Eisenbaum, H. Kaspi, On permanental processes. Stochastic Process. Appl. **119**, no. 5, 1401-1415 (2009)

[5] R.C. Griffiths, Multivariate gamma distributions. J. Multivariate Anal. **15**, 13–20 (1984)

[6] R.C. Griffiths, R.K. Milne, A class of infinitely divisible multivariate negative binomial distributions. J. Multivariate Anal. **22**, 13–23 (1987)

[7] J. B. Hough, M. Krishnapur, Y. Peres, B. Virág, Determinantal processes and independence. Probab. Surv. **3**, 206–229 (2006)

[8] O. Macchi, The coincidence approach to stochastic point processes. Adv. in Appl. Probab. **7**, 83–122 (1975)

[9] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants. I: fermion, Poisson and boson point processes. J. Funct. Anal. **205**, no. 2, 414–463 (2003)

[10] T. Shirai, Remarks on the positivity of α-determinants. Kyushu J. Math. **61**, 169–189 (2007)

[11] A. Soshnikov, Determinantal random point fields. Russian Math. Surveys **55**, 923–975 (2000)

[12] D. Vere-Jones, Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions. New Zealand J. Math. **26**, 125-149 (1997)

[13] D. Vere-Jones, A generalization of permanents and determinants. Linear Algebra Appl. **111**, 119-124 (1988)

Université Pierre et Marie Curie
LPMA
Case courrier 188
4, place Jussieu
75252 Paris Cedex 05
France

Telecom ParisTech
LTCI
46, rue Barrault
75634 Paris Cedex 13
France